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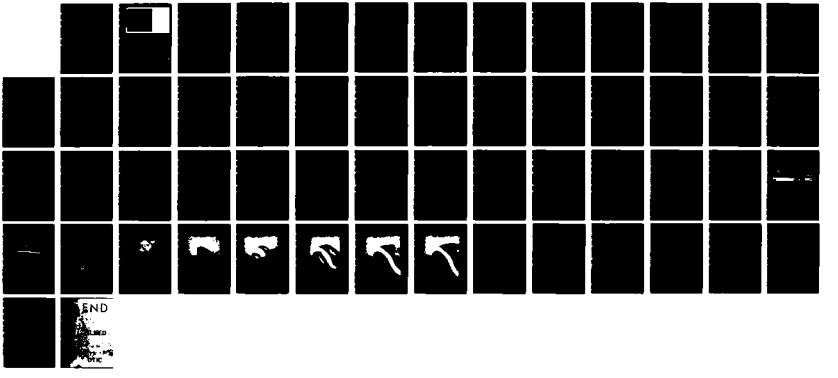

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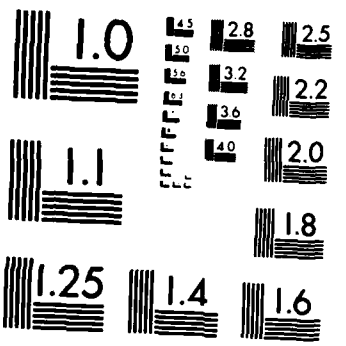

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SPLITTING METHODS FOR THE NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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ABSTRACT

Splitting methods provide efficient tools for solving linear and nonlinear time dependent problems modelled by partial differential equations. In this report we discuss the numerical solution of the Navier-Stokes equations for incompressible viscous fluids by such methods. The splitting permits decoupling the two main difficulties in the problem, namely the nonlinearity and the incompressibility. Actually these splitting methods have a broad range of applicability and can be applied for example, to the solution of eigenvalue problems.

AMS(MOS) Subject Classification: 65P05, 76D05.

Key Words: Navier-Stokes equations, operator splitting methods, nonlinear least squares, preconditioned conjugate gradient algorithms, finite element approximations, eigenvalue calculation, variational methods.

Work Unit Number 3 - Numerical Analysis and Scientific Computing.

University P.et M. Curie, Tour 55-65, 4, place Jussieu 75230 PARIS CEDEX 05, and Institut National de Recherche en Informatique et Automatique, Domaine de Voluceau, Rocquencourt, 78153 LE CHESNAY CEDEX.

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SIGNIFICANCE AND EXPLANATION

The numerical solution of the Navier-Stokes equations for incompressible viscous fluids is a complicated problem, particularly for sufficiently large Reynold numbers where strong nonlinear effects are present. Using an appropriate operator splitting method, we show in this report that it is possible to decouple the two main difficulties in the problem, namely the nonlinearlity and the incompressibility condition. This reduces the solution of the time dependent problem to a sequence of simpler stationary problems, which can be solved in many cases by standard procedures such as conjugate gradient algorithms, simple finite element approximations, etc... .

It is also shown in this report that these splitting methods can be applied to eigenvalue calculations and in fact, they have been applied elsewhere to the solution of nonlinear eigenvalue problems such as the Hartree equation in Quantum Physics.

Numerical results show the possibility of the solution methods discussed in this report when it comes to the simulation of flows closely related to practical problems.

On the basis of these numerical results and accompanying experiments it appears feasible to apply these techniques to simulate related large-scale practical problems.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

SPLITTING METHODS FOR THE NUMERICAL SOLUTION OF THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

R. Glowinski

1. Introduction. Synopsis

Solving by numerical methods the *Navier-Stokes equations*, in order to simulate *unsteady flows* of either *incompressible* or *compressible viscous fluids*, is still a challenging problem. This important problem has motivated the work of many scientists (see, e.g. TEMAM [1], GIRAULT-RAVIART [2], RAUTMANN [3], THOMASSET [4], GLOWINSKI [5] for references). Concentrating on the incompressible case we would like to show in this paper that *operator splitting methods*, like those advocated by Professor G.I. Marchuk in [6], provide quite efficient numerical schemes for solving the time dependent Navier-Stokes equations. The content of the paper is as follows :

In Section 2, we describe and comment the Navier-Stokes equations modelling unsteady flows for incompressible viscous fluids. In Section 3 we discuss some general schemes using operator splitting and apply them to the Navier-Stokes equations in order to *decouple incompressibility* and *nonlinearity* (in fact we could not resist concluding that section by showing that the same principles also apply to *eigenvalue calculations*). In Sections 4 and 5, we discuss the specific treatment of the nonlinearity and of the incompressibility, respectively. *Finite Element Approximations* are discussed in Section 6, and finally we show in Section 7 the results of several numerical experiments designed to test the methods previously discussed in the paper.

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2. Formulation of the unsteady Navier-Stokes equations for incompressible viscous fluids.

Let us consider a newtonian viscous and incompressible fluid. If Ω and Γ denote the region of the flow and its boundary, respectively, then this flow is governed by the Navier-Stokes equations

$$\frac{\partial \underset{\sim}{u}}{\partial t} - \nu \Delta \underset{\sim}{u} + (\underset{\sim}{u}.\nabla)\underset{\sim}{u} + \nabla p = \underset{\sim}{f} \quad \textit{in} \quad \Omega, \tag{2.1}$$

$$\nabla . \underset{\sim}{u} = 0 \quad \textit{in} \quad \Omega \quad \textit{(incompressibility condition)}. \tag{2.2}$$

In (2.1), (2.2) :

(i) $\underset{\sim}{u} = \{u_i\}_{i=1}^{N}$ is the *flow velocity*,

(ii) p is the *pressure*,

(iii) ν is the *viscosity* of the fluid (in normalized units we have $\nu = 1/Re$, where Re is the Reynold's number),

(iv) $\underset{\sim}{f}$ is a *density* of external forces.

In (2.1), $(\underset{\sim}{u}.\nabla)\underset{\sim}{u}$ is a symbolic notation for the nonlinear (vector) term :

$$\{ \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} \}_{i=1}^{N} .$$

Boundary conditions have to be added ; for example, in the case of the airfoil A of Fig. 2.1, below, we have (since the fluid is *viscous*) the following adherence condition

$$\underset{\sim}{u} = \underset{\sim}{0} \quad \textit{on} \quad \partial A = \Gamma_A . \tag{2.3}$$

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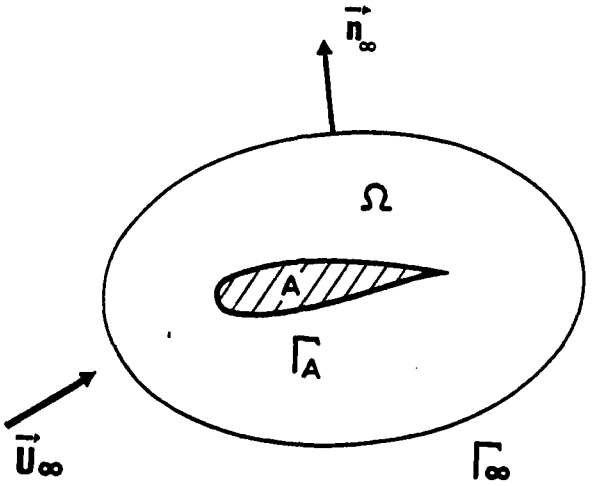


Figure 2.1.

Typical conditions at infinity are

$$\underset{\sim}{u} = \underset{\sim}{u}_\infty \quad , \tag{2.4}$$

where $\underset{\sim}{u}_\infty$ is a constant vector (with regards to the space variables at least).
Finally, for the time dependent problem (2.1), (2.2), an initial condition such as

$$\underset{\sim}{u}(x,0) = \underset{\sim}{u}_o(x) \quad on \quad \Omega \ , \tag{2.5}$$

where $\underset{\sim}{u}_o$ is given (with $\underset{\sim}{\nabla}.\underset{\sim}{u}_o = 0$), is usually prescribed.
With the methods described in this paper, we can also treat boundary conditions such as

$$\underset{\sim}{u} = \underset{\sim}{g}_o \quad on \quad \Gamma_o \ , \quad \nu \frac{\partial \underset{\sim}{u}}{\partial n} - \underset{\sim}{n}\, p = \underset{\sim}{g}_1 \quad on \ \Gamma_1 \ , \tag{2.6}$$

where Γ_o, Γ_1 are subsets of Γ such that $\Gamma_o \cap \Gamma_1 = \emptyset$, $\Gamma_o \cup \Gamma_1 = \Gamma$, and where $\underset{\sim}{n}$ is the outward unit normal vector at Γ (see [5, Appendix 3]

for the numerical implementation of (2.6)).
In two dimensions, it may be convenient to formulate the Navier-Stokes equations using a *stream-function-vorticity* formulation (see, e.g., FORTIN-THOMASSET [7], GIRAULT-RAVIART [2], GLOWINSKI-PIRONNEAU [8], REINHART [9], KELLER-SCHREIBER [10], GLOWINSKI-KELLER-REINHART [11], ROACHE [12]).

Solving the above Navier-Stokes equations, even at moderate Reynold number (say $Re \simeq 10^3$), is a non trivial task because the following difficulties

(i) It is a *nonlinear* system of partial differential equations,

(ii) We have the *incompressibility* condition (2.2).
In the following Section 3, we shall see how *time discretization* by *operator splitting* methods can decouple the *nonlinear* and *incompressibility* difficulties.

To conclude this section, let us mention that a mathematical analysis of the Navier-Stokes equations for incompressible viscous fluids can be found in e.g. LIONS [13], LADYSHENSKAYA [14], TEMAM [1] and TARTAR [15].

3. Time discretization by operator splitting methods. Applications.

3.1. Generalities. Description of the basic schemes.

Let us consider a real *Hilbert space* H ; we consider in H the following *initial value problem*

$$(3.1)\qquad \begin{cases} \dfrac{du}{dt} + A(u) = f \ , \\ \\ u(0) = u_o \ , \end{cases}$$

where A is an operator from H to H about which we are not very specific and where f is a *source term* and u_o the *initial value*, respectively. We suppose that A admits a *non trivial decomposition*

(3.2) $\quad A = A_1 + A_2$

(*non trivial* meaning that A_1 and A_2 are individually simpler than A). With Δt (> 0) a *time discretization* step, let us define several schemes taking advantage of the decomposition (3.2) :

A. A Peaceman-Rachford time discretization scheme.

The scheme is defined as follows

(3.3) $\quad u^0 = u_0$,

then for $n \geq 0$, *with* u^n *known, we compute successively* $u^{n+1/2}$ *and then* u^{n+1} *by*

$$(3.4) \qquad \frac{u^{n+1/2} - u^n}{\Delta t/2} + A_1(u^{n+1/2}) + A_2(u^n) = f^{n+1/2} ,$$

$$(3.5) \qquad \frac{u^{n+1} - u^{n+1/2}}{\Delta t/2} + A_1(u^{n+1/2}) + A_2(u^{n+1}) = f^{n+1} .$$

In (3.4), (3.5) $u^{n+\alpha}$ denotes an approximation of $u((n+\alpha)\Delta t)$, and $f^{n+\alpha} = f((n+\alpha)\Delta t)$.

B. A Douglas-Rachford time discretization scheme

It is the variant of the above scheme described by

(3.6) $\quad u^0 = u_0$,

then, for $n \geq 0$, *define* $\hat{u}^{n+1}$ *and* u^{n+1} *from* u^n *by*

$$(3.7) \qquad \frac{\hat{u}^{n+1} - u^n}{\Delta t} + A_1(\hat{u}^{n+1}) + A_2(u^n) = f^{n+1} ,$$

$$(3.8) \qquad \frac{u^{n+1} - u^n}{\Delta t} + A_1(\hat{u}^{n+1}) + A_2(u^{n+1}) = f^{n+1} .$$

C. A three stages operator splitting scheme.

Let θ belongs to the open interval (0,1/2) ; the idea behind the scheme is to split the time interval $[n\Delta t,(n+1)\Delta t]$ in three subintervals, as shown on Figure 3.1,

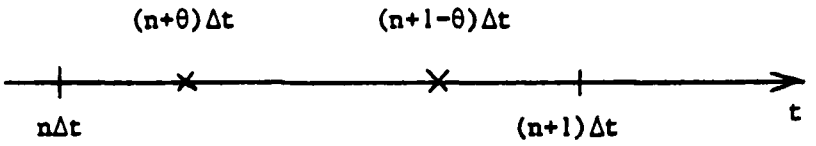


Figure 3.1.

and integrate in time using an *implicit* scheme for A_1 (resp. an *explicit* scheme for A_2) on $[n\Delta t, (n+\theta)\Delta t]$, then switch the role of A_1 and A_2 on $[(n+\theta)\Delta t, (n+1-\theta)\Delta t]$, and on $[(n+1-\theta)\Delta t, (n+1)\Delta t]$ do like on $[n\Delta t, (n+\theta)\Delta t]$. Using these principles, we obtain the following scheme, some forms of which have been advocated by STRANG [16], BEALE-MAJDA [17], LEVEQUE [18], LEVEQUE-OLIGER [19] (for $\theta = 1/4$) :

$$u^0 = u_0, \tag{3.9}$$

then, for $n \geq 0$, *we obtain* $u^{n+\theta}$, $u^{n+1-\theta}$, u^{n+1}, *from* u^n, *as follows*

$$\frac{u^{n+\theta} - u^n}{\theta\Delta t} + A_1(u^{n+\theta}) + A_2(u^n) = f^{n+\theta}, \tag{3.10}$$

$$\frac{u^{n+1-\theta} - u^{n+\theta}}{(1-2\theta)\Delta t} + A_1(u^{n+\theta}) + A_2(u^{n+1-\theta}) = f^{n+1-\theta}, \tag{3.11}$$

$$\frac{u^{n+1} - u^{n+1-\theta}}{\theta\Delta t} + A_1(u^{n+1}) + A_2(u^{n+1-\theta}) = f^{n+1}. \tag{3.12}$$

□

The convergence of (3.3)-(3.5) and (3.6)-(3.8) has been proved in LIONS-MERCIER [20] (see also GODLEWSKY [21]) under quite general *monotonicity* assumptions on A_1 and A_2 ; it is very likely that the methods used in [20] to prove the convergence of (3.3)-(3.5) and (3.6)-(3.8) still apply to (3.9)-(3.12).

3.2. Convergence and stability properties of the basic schemes

Following the approach used in [6], we shall consider for simplicity the case where $H = \mathbb{R}^N$, $f = 0$, $u_0 \in \mathbb{R}^N$, A is an $N \times N$, *symmetric* and *positive definite* matrix and where

$$A_1 = \alpha A, \quad A_2 = \beta A, \text{ with } \alpha + \beta = 1, 0 < \alpha, \beta < 1. \tag{3.13}$$

In that case the solution of (3.1) is clearly given by

$$u(t) = e^{-tA} u_o \ . \tag{3.14}$$

Analysis of scheme (3.3)-(3.5) (see also [6]) :
We have, from (3.4), (3.5), (3.13),

$$u^{n+1} = (I+ \beta \frac{\Delta t}{2} A)^{-1}(I- \alpha \frac{\Delta t}{2} A)(I+ \alpha \frac{\Delta t}{2} A)^{-1}(I-\beta \frac{\Delta t}{2} A)u^n \ . \tag{3.15}$$

Using a vector basis consisting of eigenvectors of A, we have from (3.15), with obvious notation,

$$u_i^{n+1} = \frac{(1- \alpha \frac{\Delta t}{2} \lambda_i)(1- \beta \frac{\Delta t}{2} \lambda_i)}{(1+ \alpha \frac{\Delta t}{2} \lambda_i)(1+ \beta \frac{\Delta t}{2} \lambda_i)} u_i^n \ , \tag{3.16}$$

where λ_i (> 0, $\forall i = 1,\ldots,N$) is the i^{th} eigenvalue of A ; we suppose that $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_N$. Consider now the *rational function* R_1 defined by

$$R_1(x) = \frac{(1- \frac{\alpha}{2} x)(1- \frac{\beta}{2} x)}{(1+ \frac{\alpha}{2} x)(1+ \frac{\beta}{2} x)} \ ; \tag{3.17}$$

we observe that $|R_1(x)| < 1 \quad \forall x > 0$, implying, in that simple case, the *unconditional stability* of scheme (3.3)-(3.5). Since

$$\lim_{x \to +\infty} R_1(x) = 1, \tag{3.18}$$

we observe that for stiff problems, i.e. problems such that $\lambda_N/\lambda_1 >> 1$, scheme (3.3)-(3.5) is not very good to damp, simultaneously, the components of u^n associated to the large and to the small eigenvalues of A ; from this observation, we can expect that scheme (3.3)-(3.5) is not well suited to capture the steady state solutions of stiff problems (like those obtained from the discretization of partial differential equations) ; this has been confirmed by numerical experiments.
Since

$$e^{-x} = 1 - x + \frac{x^2}{2} + x^2 \varepsilon(x) \tag{3.19}$$

and, from (3.17),

$$(3.20) \qquad R_1(x) = 1-x+\frac{x^2}{2}+x^2\,\eta(x),$$

with $\lim_{x\to 0}\varepsilon(x) = \lim_{x\to 0}\eta(x) = 0$, we have that scheme (4.3)-(4.5) is *second order accurate* in the simple case that we have considered. We observe, from (3.15) that if one takes $\alpha = \beta = 1/2$, then the two linear systems which have to be solved, at each full step, are in fact associated to the same matrix $I + \frac{\Delta t}{4} A$.

Analysis of scheme (3.6)-(3.8) : We have this time

$$(3.21) \qquad u^{n+1} = (I+\alpha\Delta t A)^{-1}(I+\beta\Delta t A)^{-1}(I+\alpha\beta|\Delta t|^2 A^2)u^n ,$$

which implies

$$(3.22) \qquad u_i^{n+1} = \frac{1+\alpha\beta|\Delta t|^2\lambda_i^2}{(1+\alpha\Delta t\,\lambda_i)(1+\beta\Delta t\,\lambda_i)}\,u_i^n .$$

If we define now R_2 by

$$(3.23) \qquad R_2(x) = \frac{1+\alpha\beta x^2}{(1+\alpha x)(1+\beta x)},$$

we observe that $0 < R_2(x) < 1 \quad \forall x > 0$, implying in turn, in the case under discussion, the *unconditional stability* of scheme (3.6)-(3.8). Since we have again

$$(3.24) \qquad \lim_{x\to+\infty} R_2(x) = 1,$$

scheme (3.6)-(3.8) may behave poorly for stiff problems and be not too efficient for capturing steady state solutions. About the accuracy of scheme (3.6)-(3.8), we should easily prove that

$$(3.25) \qquad R_2(x) = 1-x+x^2+x^2\,\eta(x),$$

with $\lim_{x\to 0}\eta(x) = 0$, implying if we compare to (3.19), that scheme (3.6)-(3.8) is only *first order accurate*.

The good choice for α and β is again $\alpha = \beta = 1/2$.

Analysis of scheme (3.9)-(3.12): We have (with $\theta' = 1-2\theta$)

$$(3.26)\qquad \begin{cases} u^{n+1} = (I+\alpha\theta\Delta t\, A)^{-1}(I-\beta\theta\Delta t\, A)(I+\beta\theta'\Delta t\, A)^{-1}(I-\alpha\theta'\Delta t A) \\ \quad \times (I+\alpha\theta\,\Delta t\, A)^{-1}(I-\beta\theta\,\Delta t\, A)u^{n}\,, \end{cases}$$

which implies

$$(3.27)\qquad u_i^{n+1} = \frac{(1-\beta\theta\Delta t\,\lambda_i)^2(1-\alpha\theta'\Delta t\,\lambda_i)}{(1+\alpha\theta\Delta t\,\lambda_i)^2(1+\beta\theta'\Delta t\,\lambda_i)}\, u_i^n\,.$$

Consider now the rational function R_3 defined by

$$(3.28)\qquad R_3(x) = \frac{(1-\beta\theta x)^2(1-\alpha\theta' x)}{(1+\alpha\theta x)^2(1+\beta\theta' x)}\,.$$

Since

$$(3.29)\qquad \lim_{x\to+\infty} |R_3(x)| = \beta/\alpha\,,$$

we should prescribe

$$(3.30)\qquad \alpha \geq \beta$$

to have, from (3.26), (3.27), the stability of scheme (3.9)-(3.12) for the *large eigenvalues* of A. We discuss now the accuracy of scheme (3.9)-(3.12) ; we can show that

$$(3.31)\qquad R_3(x) = 1-x+\frac{x^2}{2}\{1+(\beta^2-\alpha^2)(2\theta^2-4\theta+1)\} + x^2\,\eta(x),$$

with $\lim_{x\to 0} \eta(x) = 0$. It follows from (3.31) that scheme (3.9)-(3.12) is *second order accurate* if, either

$$(3.32)\qquad \alpha = \beta\ (= 1/2\ \textit{from}\ (3.13))\,,$$

or

(3.33) $\theta = 1 - \sqrt{2}/2 = .29289....;$

scheme (3.9)-(3.12) is only *first order accurate* if neither (3.32), nor (3.33) holds.

If one takes $\alpha = \beta = 1/2$ it follows from (3.27) that scheme (3.9)-(3.12) is *unconditionnaly stable* $\forall\theta \in]0,1/2[$; however since (from (3.29)) we have

(3.34) $$\lim_{x \to +\infty} |R_3(x)| = 1$$

the remark done for scheme (3.3)-(3.5), about the integration of stiff systems, still holds. In general, we shall choose α and β in order to have the same matrix for all the partial steps of the integration method, i.e. α,β,θ have to satisfy

(3.35) $\alpha\theta = \beta(1-2\theta)$

which implies

(3.36) $\alpha = (1-2\theta)/(1-\theta),\ \beta = \theta/(1-\theta).$

Combining (3.30), (3.36) we obtain

(3.37) $0 < \theta \leq 1/3.$

For $\theta = 1/3$, (3.36) implies $\alpha = \beta = 1/2$.

If $0 < \theta < 1/3$ and if α and β are given by (3.36) we have then

(3.38) $$\lim_{n \to +\infty} |R_3(x)| = \frac{\beta}{\alpha} = \frac{\theta}{1-2\theta} < 1 .$$

Actually we can prove that $\theta \in]0,1/3\ ]$ and α and β given by (3.36), imply the *unconditional stability* of scheme (3.9)-(3.12) ; moreover if $\theta \in]0,1/3\ [$ (with α, β still given by (3.36)), property (3.38) makes that scheme (3.9)-(3.12) has good asymptotic properties as $n \to +\infty$ and for example is well suited to compute steady state solutions.

If $\theta = 1 - \sqrt{2}/2$ (resp.$\theta = .25$), we have $\alpha = 2-\sqrt{2}$, $\beta = \sqrt{2}-1$, $\beta/\alpha = 1/\sqrt{2}$ (resp. $\alpha = 2/3$, $\beta = 1/3$, $\beta/\alpha = 1/2$).

3.3. Application to the solution of the time dependent Navier-Stokes equations.

We discuss now the application of the schemes described in Sec. 3.1 to the solution of the time dependent Navier-Stokes equations (2.1), (2.2) with the initial value condition (2.5) ; we suppose for simplicity that the boundary conditions are of the Dirichlet type, i.e.

$$\underset{\sim}{u} = \underset{\sim}{g} \quad on \quad \Gamma \quad (with \int_{\Gamma} \underset{\sim}{g}.\underset{\sim}{n}\, d\Gamma = 0). \tag{3.39}$$

3.3.1. A first operator splitting method.

This method, which is directly derived from the *Peaceman-Rachford scheme* (3.3)-(3.5) is described as follows :

$$\underset{\sim}{u}^{0} = \underset{\sim}{u}_{0}, \tag{3.40}$$

then for $n \geq 0$ *and starting from* $\underset{\sim}{u}^{n}$ *we compute* $\{\underset{\sim}{u}^{n+1/2}, p^{n+1/2}\}$ *and* $\underset{\sim}{u}^{n+1}$ *by solving*

$$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1/2} - \underset{\sim}{u}^{n}}{\Delta t/2} - \dfrac{\nu}{2}\,\Delta\underset{\sim}{u}^{n+1/2} + \nabla p^{n+1/2} = \underset{\sim}{f}^{n+1/2} + \dfrac{\nu}{2}\,\Delta\underset{\sim}{u}^{n} - (\underset{\sim}{u}^{n}.\nabla)\underset{\sim}{u}^{n} \ in \\ \nabla.\underset{\sim}{u}^{n+1/2} = 0 \quad in \quad \Omega\,, \\ \underset{\sim}{u}^{n+1/2} = \underset{\sim}{g}^{n+1/2} \ on \ \Gamma\,, \end{cases} \tag{3.41}$$

and

$$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^{n+1/2}}{\Delta t/2} - \dfrac{\nu}{2}\,\Delta\underset{\sim}{u}^{n+1} + (\underset{\sim}{u}^{n+1}.\nabla)\underset{\sim}{u}^{n+1} = \underset{\sim}{f}^{n+1} + \dfrac{\nu}{2}\,\Delta\underset{\sim}{u}^{n+1/2} - \nabla p^{n+1/2}\ in \\ \underset{\sim}{u}^{n+1} = \underset{\sim}{g}^{n+1} \quad on \ \Gamma\,, \end{cases} \tag{3.42}$$

respectively.

3.3.2. A second operator splitting method

This method is derived from scheme (3.9)-(3.12) and is described as follows :

$$\underset{\sim}{u}^{0} = \underset{\sim}{u}_{o} \,, \tag{3.43}$$

then for $n \geq 0$ *and starting from* $\underset{\sim}{u}^{n}$ *we solve*

$$\begin{cases} \dfrac{\underset{\sim}{u}^{n+\theta} - \underset{\sim}{u}^{n}}{\theta\Delta t} - \alpha\nu\Delta\underset{\sim}{u}^{n+\theta} + \underset{\sim}{\nabla} p^{n+\theta} = \underset{\sim}{f}^{n+\theta} + \beta\nu\Delta\underset{\sim}{u}^{n} - (\underset{\sim}{u}^{n}.\underset{\sim}{\nabla})\underset{\sim}{u}^{n} \ \text{in } \Omega \,, \\ \underset{\sim}{\nabla}.\underset{\sim}{u}^{n+\theta} = 0 \ \text{in } \Omega, \\ \underset{\sim}{u}^{n+\theta} = \underset{\sim}{g}^{n+\theta} \ \text{on } \Gamma, \end{cases} \tag{3.44}$$

$$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1-\theta} - \underset{\sim}{u}^{n+\theta}}{(1-2\theta)\Delta t} - \beta\nu\Delta\underset{\sim}{u}^{n+1-\theta} + (\underset{\sim}{u}^{n+1-\theta}.\underset{\sim}{\nabla})\underset{\sim}{u}^{n+1-\theta} = \\ \underset{\sim}{f}^{n+1-\theta} + \alpha\nu\Delta\underset{\sim}{u}^{n+\theta} - \underset{\sim}{\nabla} p^{n+\theta} \ \text{in } \Omega \,, \\ \underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{g}^{n+1-\theta} \ \text{on } \Gamma \,, \end{cases} \tag{3.45}$$

$$\begin{cases} \dfrac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^{n+1-\theta}}{\theta\Delta t} - \alpha\nu\Delta\underset{\sim}{u}^{n+1} + \underset{\sim}{\nabla} p^{n+1} = f^{n+1} + \beta\nu\Delta\underset{\sim}{u}^{n+1-\theta} - \\ \qquad (\underset{\sim}{u}^{n+1-\theta}.\underset{\sim}{\nabla})\underset{\sim}{u}^{n+1-\theta} \ \text{in } \Omega \,, \\ \underset{\sim}{\nabla}.\underset{\sim}{u}^{n+1} = 0 \ \text{in } \Omega, \\ \underset{\sim}{u}^{n+1} = \underset{\sim}{g}^{n+1} \ \text{on } \Gamma \,. \end{cases} \tag{3.46}$$

For the choice of α and β, see Sec. 3.3.3, below.

3.3.3. Some comments and remarks concerning schemes (3.40)-(3.42) and (3.43)-(3.46).

Using the two above operator splitting methods we have been able to decouple *nonlinearity* and *incompressibility* in the Navier-Stokes equations (2.1), (2.2). We shall describe -briefly- in the following Sections 4 and 5 the specific treatment of the subproblems encountered at each step of (3.40)-(3.42) and (3.43)-(3.46).

We observe that $\underset{\sim}{u}^{n+1/2}$ and $\underset{\sim}{u}^{n+\theta}$, $\underset{\sim}{u}^{n+1}$ are obtained from the solution of *linear problems* very close to the *steady Stokes problem*. Despite its

greater complexity scheme (3.43)-(3.46) is (per step) almost as economical to use as scheme (3.40)-(3.42) ; this is mainly due to the fact that the "quasi" steady Stokes problems (3.41) and (3.44), (3.46) (actually convenient finite element approximation of them) can be solved by quite efficient solvers so that most of the computer time used to solve a full step is in fact used to solve the nonlinear subproblem. The good choice for α and β is given by (3.36) if one uses scheme (3.43)-(3.46) ; with such a choice many computer subprograms can be used for both the linear and nonlinear subproblems, resulting therefore in quite substantial core memory savings.

3.4. Application to eigenvalue calculations.

3.4.1. Generalities. Synopsis.

The main goal of this section is to show that the concepts introduced in Sec. 3.1 apply also to *eigenvalue calculations* at least for *symmetric* matrices (or operator). Since the resulting methods belong to the class of the so-called *inverse power methods*, the new approach brings very little to the *linear eigenvalue problem* (for which a basic reference is [23]), but it is nicely suited to solve *nonlinear eigenvalue problems* like for example the *Hartree equation* in Quantum Physics (see [24] , [25] for more details).

3.4.2. Formulation of the problem.

Let $\underset{\sim}{A}$ be a *symmetric* $N \times N$, real matrix ; we denote by λ_i the eigenvalues of $\underset{\sim}{A}$ and we suppose that $\lambda_1 \leq \lambda_2 \leq \ldots \lambda_N$.
We concentrate on the calculation of the smallest eigenvalue of $\underset{\sim}{A}$ (i.e. λ_1). it is well known that λ_1 satisfies

$$\lambda_1 = \min_{\underset{\sim}{v} \in S} (\underset{\sim}{A}\underset{\sim}{v}, \underset{\sim}{v}), \tag{3.47}$$

where

$$S = \{\underset{\sim}{v} \mid \underset{\sim}{v} \in \mathbb{R}^N, \ \|\underset{\sim}{v}\| = 1\} , \tag{3.48}$$

$$\begin{cases} (\underset{\sim}{v}, \underset{\sim}{w}) = \sum_{i=1}^{N} v_i w_i, \ \underset{\sim}{v} = \{v_i\}_{i=1}^{N}, \ \underset{\sim}{w} = \{w_i\}_{i=1}^{N} , \\ \|\underset{\sim}{v}\| = (\underset{\sim}{v}, \underset{\sim}{v})^{1/2} . \end{cases} \tag{3.49}$$

Let introduce now the functional I_S defined by

$$(3.50) \qquad I_S(\underset{\sim}{v}) = \begin{cases} 0 & \text{if } \underset{\sim}{v} \in S, \\ +\infty & \text{if } \underset{\sim}{v} \in S ; \end{cases}$$

I_S is the *indicator functional* of S. We clearly have

$$(3.51) \qquad \lambda_1/2 = \min_{\underset{\sim}{v} \in \mathbb{R}^N} \{\tfrac{1}{2}(\underset{\sim\sim}{A}\underset{\sim}{v}, \underset{\sim}{v}) + I_S(\underset{\sim}{v})\} .$$

Suppose that I_S is *differentiable* (which is definitely not the case) and denote by ∂I_S its differential ; if $\underset{\sim}{u}$ is a *minimizer* in problems (3.47), (3.51), i.e. an *eigenvector* of norm one, associated to λ_1, $\underset{\sim}{u}$ satisfies

$$(3.52) \qquad \underset{\sim\sim}{A}\underset{\sim}{u} + \partial I_S(\underset{\sim}{u}) = \underset{\sim}{0} .$$

It is then quite natural to associate to the nonlinear "equation" (3.52) the initial value problem below

$$(3.53) \qquad \begin{cases} \dfrac{d\underset{\sim}{u}}{dt} + \underset{\sim\sim}{A}\underset{\sim}{u} + \partial I_S(\underset{\sim}{u}) = \underset{\sim}{0} , \\ \underset{\sim}{u}(0) = \underset{\sim}{u}_o , \end{cases}$$

and to look for the *steady state solutions* of (3.53), i.e. to $\lim_{t \to +\infty} \underset{\sim}{u}(t)$, if such a limit exists. From the special form of (3.53), it is tempting to solve it using the *operator splitting* methods discussed in Secs. 3.1, 3.2. The resulting algorithms are described in the following Sec. 3.4.3.

3.4.3. Solution of (3.52), via (3.53), by operator splitting methods.

We apply now the operator splitting methods of Sec. 3.1 with $A_1 = \partial I_S$, $A_2 = \underset{\sim\sim}{A}$ and $f = \underset{\sim}{0}$.

Application of the Peaceman-Rachford scheme (3.3)-(3.5) :

$$(3.54) \qquad \underset{\sim}{u}^o = \underset{\sim}{u}_o ,$$

then for $n \geq 0$ *with* $\underset{\sim}{u}^n$ *known we compute successively* $\underset{\sim}{u}^{n+1/2}$ *and* $\underset{\sim}{u}^{n+1}$ *by*

$$\frac{\underset{\sim}{u}^{n+1/2} - \underset{\sim}{u}^n}{\Delta t/2} + \partial I_S(\underset{\sim}{u}^{n+1/2}) + \underset{\sim}{A}\,\underset{\sim}{u}^n = \underset{\sim}{0}\,, \tag{3.55}$$

$$\frac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^{n+1/2}}{\Delta t/2} + \partial I_S(\underset{\sim}{u}^{n+1/2}) + \underset{\sim}{A}\,\underset{\sim}{u}^{n+1} = \underset{\sim}{0}\,. \tag{3.56}$$

Application of the Douglas-Rachford scheme (3.6)-(3.8) :

$$\underset{\sim}{u}^0 = \underset{\sim}{u}_0\,, \tag{3.57}$$

then for $n \geq 0$, *with* $\underset{\sim}{u}^n$ *known we compute successively* $\underset{\sim}{\hat{u}}^{n+1}$ *and* $\underset{\sim}{u}^{n+1}$ *by*

$$\frac{\underset{\sim}{\hat{u}}^{n+1} - \underset{\sim}{u}^n}{\Delta t} + \partial I_S(\underset{\sim}{\hat{u}}^{n+1}) + \underset{\sim}{A}\,\underset{\sim}{u}^n = \underset{\sim}{0}, \tag{3.58}$$

$$\frac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^n}{\Delta t} + \partial I_S(\underset{\sim}{\hat{u}}^{n+1}) + \underset{\sim}{A}\,\underset{\sim}{u}^{n+1} = \underset{\sim}{0}\,. \tag{3.59}$$

Application of the three stages scheme (3.9)-(3.12) :

$$\underset{\sim}{u}^0 = \underset{\sim}{u}_0\,, \tag{3.60}$$

then for $n \geq 0$, *with* $\underset{\sim}{u}^n$ *known we compute successively* $\underset{\sim}{u}^{n+\theta}$, $\underset{\sim}{u}^{n+1-\theta}$ *and* $\underset{\sim}{u}^{n+1}$ *by*

$$\frac{\underset{\sim}{u}^{n+\theta} - \underset{\sim}{u}^n}{\theta\Delta t} + \partial I_S(\underset{\sim}{u}^{n+\theta}) + \underset{\sim}{A}\,\underset{\sim}{u}^n = \underset{\sim}{0}, \tag{3.61}$$

$$\frac{\underset{\sim}{u}^{n+1-\theta} - \underset{\sim}{u}^{n+\theta}}{(1-2\theta)\,\Delta t} + \partial I_S(\underset{\sim}{u}^{n+\theta}) + \underset{\sim}{A}\,\underset{\sim}{u}^{n+1-\theta} = \underset{\sim}{0}, \tag{3.62}$$

$$\frac{\underset{\sim}{u}^{n+1} - \underset{\sim}{u}^{n+1-\theta}}{\theta\Delta t} + \partial I_S(\underset{\sim}{u}^{n+1}) + \underset{\sim}{A}\,\underset{\sim}{u}^{n+1} = \underset{\sim}{0}\,. \tag{3.63}$$

In order to derive more practical formulations of the above algorithms, let consider algorithm (3.54)-(3.56) (the conclusion to be obtained will hold also for the two other methods) :

We observe that (3.55) is in fact a *necessary condition* of optimality, for the following minimization problem

$$(3.64)\quad \begin{cases} \textit{Find } \underset{\sim}{u}^{n+1/2} \in S \textit{ such that} \\ \\ J_n(\underset{\sim}{u}^{n+1/2}) \le J_n(\underset{\sim}{v}),\ \forall \underset{\sim}{v} \in S, \end{cases}$$

with

$$J_n(\underset{\sim}{v}) = \frac{1}{2}\|\underset{\sim}{v}\|^2 - (\underset{\sim}{u}^n - \frac{\Delta t}{2}\underset{\sim}{A}\,\underset{\sim}{u}^n,\ \underset{\sim}{v})\ .$$

Since $\|\underset{\sim}{v}\| = 1$, $\forall \underset{\sim}{v} \in S$, the solution of (3.64) is given by

$$\underset{\sim}{u}^{n+1/2} = \frac{\underset{\sim}{u}^n - \frac{\Delta t}{2}\underset{\sim}{A}\,\underset{\sim}{u}^n}{\|\underset{\sim}{u}^n - \frac{\Delta t}{2}\underset{\sim}{A}\,\underset{\sim}{u}^n\|}\ .$$

On the other hand it follows from (3.55) that

$$\frac{\Delta t}{2}\,\partial I_S(\underset{\sim}{u}^{n+1/2}) = \underset{\sim}{u}^n - \frac{\Delta t}{2}\underset{\sim}{A}\,\underset{\sim}{u}^n - \underset{\sim}{u}^{n+1/2}\ .$$

which, combined to (3.56), implies

$$(\underset{\sim}{I} + \frac{\Delta t}{2}\underset{\sim}{A})\underset{\sim}{u}^{n+1} = 2\underset{\sim}{u}^{n+1/2} - (\underset{\sim}{u}^n - \frac{\Delta t}{2}\underset{\sim}{A}\,\underset{\sim}{u}^n).$$

Collecting the above results we obtain the following practical formulation for algorithm (3.54)-(3.56).

Practical formulation of algorithm (3.54)-(3.56) :

$$(3.65)\quad \underset{\sim}{u}^0 = \underset{\sim}{u}_0,$$

then for $n \ge 0$, *compute* $\underset{\sim}{p}^n$ *and* $\underset{\sim}{u}^{n+1}$, *from* $\underset{\sim}{u}^n$, *by*

$$(3.66)\quad \underset{\sim}{p}^n = \underset{\sim}{u}^n - \frac{\Delta t}{2}\underset{\sim}{A}\,\underset{\sim}{u}^n\ ,$$

$$(3.67)\quad \underset{\sim}{u}^{n+1} = (I + \frac{\Delta t}{2}\underset{\sim}{A})^{-1}(\frac{2}{\|p^n\|} - 1)\ \underset{\sim}{p}^n\ .\qquad \square$$

Using similar calculations we should obtain the following formulations for (3.57)-(3.59) and (3.60)-(3.63) :

Practical formulation of algorithm (3.57)-(3.59) :

$$\underset{\sim}{u}^0 = \underset{\sim}{u}_0, \tag{3.68}$$

then for $n \geq 0$, *compute* $\underset{\sim}{p}^n$ *and* $\underset{\sim}{u}^{n+1}$, *from* $\underset{\sim}{u}^n$, *by*

$$\underset{\sim}{p}^n = \underset{\sim}{u}^n - \Delta t\, \underset{\sim}{A}\, \underset{\sim}{u}^n , \tag{3.69}$$

$$\underset{\sim}{u}^{n+1} = (\underset{\sim}{I} + \Delta t\, \underset{\sim}{A})^{-1} \{\underset{\sim}{u}^n + (\frac{1}{\|\underset{\sim}{p}^n\|} - 1)\underset{\sim}{p}^n\} . \tag{3.70}$$

Practical formulation of algorithm (3.60)-(3.63) :

$$\underset{\sim}{u}^0 = \underset{\sim}{u}_0 , \tag{3.71}$$

then for $n \geq 0$, *compute* $\underset{\sim}{p}^n$, $\underset{\sim}{u}^{n+\theta}$, $\underset{\sim}{u}^{n+1-\theta}$, $\underset{\sim}{p}^{n+1-\theta}$, $\underset{\sim}{u}^{n+1}$, *from* $\underset{\sim}{u}^n$ *by*

$$\underset{\sim}{p}^n = \underset{\sim}{u}^n - \theta\Delta t\, \underset{\sim}{A}\, \underset{\sim}{u}^n , \tag{3.72}$$

$$\underset{\sim}{u}^{n+\theta} = \frac{\underset{\sim}{p}^n}{\|\underset{\sim}{p}^n\|} , \tag{3.73}$$

$$\left\{ \begin{array}{l} \underset{\sim}{u}^{n+1-\theta} = (\underset{\sim}{I} + (1-2\theta)\Delta t\, \underset{\sim}{A})^{-1} (\frac{1-\theta}{\theta}\, \underset{\sim}{u}^{n+\theta} - \frac{1-2\theta}{\theta}\, \underset{\sim}{p}^n) = \\ \qquad = (\underset{\sim}{I} + (1-2\theta)\Delta t\, \underset{\sim}{A})^{-1} (\frac{1-\theta}{\theta} \frac{1}{\|\underset{\sim}{p}^n\|} - \frac{1-2\theta}{\theta})\underset{\sim}{p}^n , \end{array} \right. \tag{3.74}$$

$$\underset{\sim}{p}^{n+1-\theta} = \underset{\sim}{u}^{n+1-\theta} - \theta\Delta t\, \underset{\sim}{A}\, \underset{\sim}{u}^{n+1-\theta} , \tag{3.75}$$

$$\underset{\sim}{u}^{n+1} = \frac{\underset{\sim}{p}^{n+1-\theta}}{\|\underset{\sim}{p}^{n+1-\theta}\|} . \quad \square \tag{3.76}$$

From the above relations the three algorithms which have been described appear as variations of the well-known *inverse power methods* (see, e.g., [23] for a detailed analysis of such methods). From our numerical experiments the various vector sequences generated by the above algorithms converge *linearly* to an eigenvector associated to the smallest eigenvalue λ_1, while $(\underset{\sim}{A}\, \underset{\sim}{u}^n, \underset{\sim}{u}^n)$ converges *quadratically* to λ_1. Actually the conver-

gence is still good if λ_1 is a *multiple* eigenvalue. It is interesting to observe that the fastest algorithm is (3.71)-(3.76), then (3.65)-(3.67), then finally (3.68)-(3.70) ; these results agree with the analysis done for the trivial linear problem that we considered in Sec. 3.2.

In the following sections we go back to the Navier-Stokes equations and their numerical treatment.

4. Least squares conjugate gradient solution of the nonlinear subproblems obtained from the time discretization of the Navier-Stokes Equations by the methods of Section 3.3.

4.1. Classical and variational formulation. Synopsis

At each full step of the operator splitting methods (3.40)-(3.42) and (3.43)-(3.46) we have to solve a nonlinear elliptic system of the following type

$$(4.1)\qquad \begin{cases} \alpha \underset{\sim}{u} - \nu \Delta \underset{\sim}{u} + (\underset{\sim}{u}.\nabla)\underset{\sim}{u} = \underset{\sim}{f} \quad \textit{in } \Omega, \\ \underset{\sim}{u} = \underset{\sim}{g} \quad \textit{on } \Gamma, \end{cases}$$

where α and ν are two positive constants and where $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are two given functions defined on Ω and Γ , respectively. We do not discuss here the existence and uniqueness of solutions for problem (4.1).

We introduce now the following functional spaces of *Sobolev's* type

$$(4.2)\qquad H^1(\Omega) = \{\phi \mid \phi \in L^2(\Omega),\ \frac{\partial \phi}{\partial x_i} \in L^2(\Omega),\ \forall i = 1,\ldots,N\} ,$$

$$(4.3)\qquad H_0^1(\Omega) = \{\phi \mid \phi \in H^1(\Omega),\ \phi = 0 \quad \textit{on } \Gamma\} ,$$

$$(4.4)\qquad V_o = (H_0^1(\Omega))^N ,$$

$$(4.5)\qquad V_g = \{\underset{\sim}{v} \mid \underset{\sim}{v} \in (H^1(\Omega))^N,\ \underset{\sim}{v} = \underset{\sim}{g} \quad \textit{on } \Gamma\} ;$$

if $\underset{\sim}{g}$ is sufficiently smooth then V_g is *nonempty*.

We shall use the following notation

$$dx = dx_1, \ldots, dx_N$$

and if $\underset{\sim}{u} = \{u_i\}_{i=1}^N$, $\underset{\sim}{v} = \{v_i\}_{i=1}^N$, then

$$\underset{\sim}{u}.\underset{\sim}{v} = \sum_{i=1}^{N} u_i\, v_i,$$

$$\underset{\sim\sim}{\nabla u}.\underset{\sim\sim}{\nabla v} = \sum_{i=1}^{N} \underset{\sim}{\nabla u_i}.\underset{\sim}{\nabla v_i} = \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial u_i}{\partial x_j}\,\frac{\partial v_i}{\partial x_j}.$$

Using *Green's formula* we can prove that for sufficiently smooth functions $\underset{\sim}{u}$ and $\underset{\sim}{v}$ belonging to $(H^1(\Omega))^N$ and V_o, respectively, we have

$$(4.6) \qquad -\int_\Omega \Delta\underset{\sim}{u}.\underset{\sim}{v}\, dx = \int_\Omega \underset{\sim\sim}{\nabla u}.\underset{\sim\sim}{\nabla v}\, dx .$$

It can also be proved that if $\underset{\sim}{u}$ is a solution of (4.1), belonging to V_g, it is also a solution of the following *nonlinear variational problem*

$$(4.7) \qquad \begin{cases} \underset{\sim}{u} \in V_g, \\ \alpha \int_\Omega \underset{\sim}{u}.\underset{\sim}{v}\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla u}.\underset{\sim\sim}{\nabla v}\, dx + \int_\Omega ((\underset{\sim}{u}.\nabla)\underset{\sim}{u}).\underset{\sim}{v}\, dx = \int_\Omega \underset{\sim}{f}.\underset{\sim}{v}\, dx \ , \forall \underset{\sim}{v} \in V_o, \end{cases}$$

and *conversely*. We observe that (4.1),(4.7) is not equivalent to a problem of the *Calculus of Variations* since there is no functional of $\underset{\sim}{v}$ with $(\underset{\sim}{v}.\nabla)\underset{\sim}{v}$ as differential ; however using a convenient *least squares* formulation we shall be able to solve (4.1), (4.7) by iterative methods originating from *Nonlinear Programming*, such as *conjugate gradient* for example.

4.2. Least squares formulation of (4.1), (4.7).

Let $\underset{\sim}{v} \in V_g$; from $\underset{\sim}{v}$ we define $\underset{\sim}{y}$ $(= \underset{\sim}{y}(\underset{\sim}{v})) \in V_o$ as the solution of

$$(4.8) \qquad \begin{cases} \alpha\underset{\sim}{y} - \nu\Delta\underset{\sim}{y} = \alpha\underset{\sim}{v} - \nu\Delta\underset{\sim}{v} + (\underset{\sim}{v}.\nabla)\underset{\sim}{v} - \underset{\sim}{f} \quad \textit{in} \quad \Omega , \\ \underset{\sim}{y} = \underset{\sim}{0} \quad \textit{on} \quad \Gamma. \end{cases}$$

We observe that $\underset{\sim}{y}$ is obtained from $\underset{\sim}{v}$ via the solution of N uncoupled linear Poisson problems (one for each component of $\underset{\sim}{y}$), using (4.6) it can be shown that problem (4.8) is *equivalent* to the *linear variational problem*

$$(4.9)\qquad \begin{cases} \textit{Find } \underset{\sim}{y} \in V_o \textit{ such that, } \forall \underset{\sim}{z} \in V_o, \textit{ we have} \\ \alpha \int_\Omega \underset{\sim}{y}.\underset{\sim}{z}\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla y}.\underset{\sim\sim}{\nabla z}\, dx = \alpha \int_\Omega \underset{\sim}{v}.\underset{\sim}{z}\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla v}.\underset{\sim\sim}{\nabla z}\, dx \\ + \int_\Omega ((\underset{\sim}{v}.\underset{\sim}{\nabla})\underset{\sim}{v}).\underset{\sim}{z}\, dx - \int_\Omega \underset{\sim}{f}.\underset{\sim}{z}\, dx, \end{cases}$$

which has a unique solution. Suppose now that $\underset{\sim}{v}$ is a solution of the nonlinear problem (4.1), (4.7) ; the corresponding $\underset{\sim}{y}$ (obtained from the solution of (4.8), (4.9)) is clearly $\underset{\sim}{y} = \underset{\sim}{0}$; from this observation it is quite natural to introduce the following *(nonlinear) least squares formulation* of (4.1), (4.7)

$$(4.10)\qquad \begin{cases} \textit{Find } \underset{\sim}{u} \in V_g \quad \textit{such that} \\ J(\underset{\sim}{u}) \le J(\underset{\sim}{v}),\ \forall \underset{\sim}{v} \in V_g, \end{cases}$$

where $J : (H^1(\Omega))^N \to \mathbb{R}$ is the function of $\underset{\sim}{v}$ defined by

$$(4.11)\qquad J(\underset{\sim}{v}) = \frac{1}{2} \int_\Omega \{\alpha |\underset{\sim}{y}|^2 + \nu |\underset{\sim\sim}{\nabla y}|^2\}\, dx ,$$

where $\underset{\sim}{y}$ is defined from $\underset{\sim}{v}$ by (4.8), (4.9). We observe that if $\underset{\sim}{u}$ is solution of (4.1), (4.7), then it is also a solution of (4.10) such that $J(\underset{\sim}{u}) = 0$; conversely, if $\underset{\sim}{u}$ is a solution of (4.10) such that $J(\underset{\sim}{u}) = 0$, then it is also a solution of (4.1), (4.7).

4.3. Conjugate gradient solution of the least squares problem (4.10).

4.3.1. Description of the algorithm.

We use the *Polak-Ribière* version (see [26]) of the conjugate gradient method to solve the minimization problem (4.10) ; we have then (with $J'(\underset{\sim}{v})$ the differential of J at $\underset{\sim}{v}$)

Step 0 : Initialization

$$(4.12)\qquad \underset{\sim}{u}^0 \in V_g, \quad \textit{given} ;$$

we define then $\underset{\sim}{g}^0$, $\underset{\sim}{w}^0 \in V_o$ *by*

$$(4.13)\qquad \begin{cases} \underset{\sim}{g}^0 \in V_0, \\ \alpha \int_\Omega \underset{\sim}{g}^0 . \underset{\sim}{z}\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla g}^0 . \underset{\sim\sim}{\nabla z}\, dx = \langle J'(\underset{\sim}{u}^0), \underset{\sim}{z} \rangle \quad , \ \forall \underset{\sim}{z} \in V_0 \ , \end{cases}$$

$$(4.14)\qquad \underset{\sim}{w}^0 = \underset{\sim}{g}^0 \ ,$$

respectively. □

Then for $n \geq 0$, *assuming that* $\underset{\sim}{u}^n$, $\underset{\sim}{w}^n$, $\underset{\sim}{g}^n$ *are known, we obtain* $\underset{\sim}{u}^{n+1}$, $\underset{\sim}{g}^{n+1}$, $\underset{\sim}{w}^{n+1}$ *by*

Step 1 : Descent

$$(4.15)\qquad \begin{cases} \textit{Find } \lambda_n \in \mathbb{R}, \\ J(\underset{\sim}{u}^n - \lambda_n \underset{\sim}{w}^n) \leq J(\underset{\sim}{u}^n - \lambda \underset{\sim}{w}^n), \quad \forall \lambda \in \mathbb{R} \ , \end{cases}$$

$$(4.16)\qquad \underset{\sim}{u}^{n+1} = \underset{\sim}{u}^n - \lambda_n \underset{\sim}{w}^n \ .$$

Step 2 : Calculation of the new descent direction

$$(4.17)\qquad \begin{cases} \textit{Find } \underset{\sim}{g}^{n+1} \in V_0, \textit{ such that} \\ \alpha \int_\Omega \underset{\sim}{g}^{n+1} . \underset{\sim}{z}\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla g}^{n+1} . \underset{\sim\sim}{\nabla z}\, dx = \langle J'(\underset{\sim}{u}^{n+1}), \underset{\sim}{z} \rangle \ , \ \forall \underset{\sim}{z} \in V_0, \end{cases}$$

$$(4.18)\qquad \gamma_n = \frac{\alpha \int_\Omega \underset{\sim}{g}^{n+1} . (\underset{\sim}{g}^{n+1} - \underset{\sim}{g}^n)\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla g}^{n+1} . \underset{\sim\sim}{\nabla} (\underset{\sim}{g}^{n+1} - \underset{\sim}{g}^n)\, dx}{\alpha \int_\Omega |\underset{\sim}{g}^n|^2\, dx + \nu \int_\Omega |\underset{\sim\sim}{\nabla g}^n|^2\, dx} \ ,$$

$$(4.19)\qquad \underset{\sim}{w}^{n+1} = \underset{\sim}{g}^{n+1} + \gamma_n \underset{\sim}{w}^n \ . \qquad \square$$

Do $n = n+1$, *go to* (4.15).

As we shall see, applying algorithm (4.12)-(4.19), to solve the least squares problem (4.10), requires the solution at each iteration of *exactly three Dirichlet systems* (i.e. 3N scalar Dirichlet problems) associated to the elliptic operator $\alpha I - \nu\Delta$.

4.3.2. Calculation of J'.

A most important step when making use of algorithm (4.12)-(4.19) to solve problem (4.10), is the calculation of $\langle J'(\underset{\sim}{u}^{n+1}),\underset{\sim}{z}\rangle$ at each iteration ; we should easily prove (see, e.g., [5] , [22]) that $J'(\underset{\sim}{v})$ can be identified with the *linear functional* from V_o to $\mathbb{R}$, defined by

$$(4.21)\quad \left\{ \begin{array}{l} \langle J'(\underset{\sim}{v}),\underset{\sim}{z}\rangle = \alpha \int_\Omega \underset{\sim}{y}.\underset{\sim}{z}\, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla y}.\underset{\sim\sim}{\nabla z}\, dx \\ \qquad\qquad + \int_\Omega \underset{\sim}{y}.(\underset{\sim}{v}.\underset{\sim}{\nabla})\underset{\sim}{z}\, dx + \int_\Omega \underset{\sim}{y}.(\underset{\sim}{z}.\underset{\sim}{\nabla})\underset{\sim}{v}\, dx, \ \forall \underset{\sim}{z} \in V_o \end{array} \right.$$

where

$$(\underset{\sim}{v}.\underset{\sim}{\nabla})\underset{\sim}{w} = \{ \sum_{j=1}^{N} v_j \frac{\partial w_i}{\partial x_j} \}_{i=1}^{N} \ , \ \forall \underset{\sim}{v},\ \underset{\sim}{w} \ ;$$

$\langle J'(\underset{\sim}{v}),\underset{\sim}{z}\rangle$ has therefore a *purely integral representation*, which is of major importance in view of *finite element* (or *spectral*) implementations of algorithm (4.12)-(4.19). From the above results, to obtain $\langle J'(\underset{\sim}{u}^{n+1}),\underset{\sim}{z}\rangle$ we should proceed as follows :

(i) Compute $\underset{\sim}{y}^{n+1}$, associated to $\underset{\sim}{u}^{n+1}$ by (4.8), (4.9), as indicated in Sec. 4.3.3, below.

(ii) We obtain then $\langle J'(\underset{\sim}{u}^{n+1}),\underset{\sim}{z}\rangle$ by taking $\underset{\sim}{v} = \underset{\sim}{u}^{n+1}$ and $\underset{\sim}{y} = \underset{\sim}{y}^{n+1}$ in (4.21).

4.3.3. Calculation of λ_n. Further comments on algorithm (4.12)-(4.19).

A problem of practical importance is the calculation of λ_n. Let denote by $\underset{\sim}{y}^n(\lambda)$ the solution of (4.8), (4.9) associated to $\underset{\sim}{v} = \underset{\sim}{u}^n - \lambda \underset{\sim}{w}^n$; we clearly have

$$(4.22)\qquad \underset{\sim}{y}^n(0) = \underset{\sim}{y}^n,\ \underset{\sim}{y}^n(\lambda_n) = \underset{\sim}{y}^{n+1} \ ,$$

and also

$$(4.23)\qquad \underset{\sim}{y}^n(\lambda) = \underset{\sim}{y}^n - \lambda \underset{\sim}{y}_1^n + \lambda^2\, \underset{\sim}{y}_2^n \ ,$$

where $\underset{\sim}{y}_1^n$, $\underset{\sim}{y}_2^n$ are the solutions of

$$(4.24)\qquad \begin{cases} \alpha \underset{\sim}{y}_1^n - \nu\Delta \underset{\sim}{y}_1^n = \alpha \underset{\sim}{w}^n - \nu\Delta \underset{\sim}{w}^n + (\underset{\sim}{u}^n.\underset{\sim}{\nabla})\underset{\sim}{w}^n + (\underset{\sim}{w}^n.\underset{\sim}{\nabla})\underset{\sim}{u}^n \quad \textit{in} \quad \Omega, \\ \underset{\sim}{y}_1^n = \underset{\sim}{0} \quad \textit{on} \quad \Gamma, \end{cases}$$

$$(4.25)\qquad \begin{cases} \alpha \underset{\sim}{y}_2^n - \nu\Delta \underset{\sim}{y}_2^n = (\underset{\sim}{w}^n.\underset{\sim}{\nabla})\underset{\sim}{w}^n \quad \textit{in} \ \Omega, \\ \underset{\sim}{y}_2^n = \underset{\sim}{0} \quad \textit{on} \quad \Gamma, \end{cases}$$

respectively. Since

$$(4.26)\qquad J(\underset{\sim}{u}^n - \lambda \underset{\sim}{w}^n) = \frac{1}{2}\int_\Omega \{\alpha|\underset{\sim}{y}^n(\lambda)|^2 + \nu|\underset{\sim}{\nabla y}^n(\lambda)|^2\}\, dx ,$$

the function $\lambda \to J(\underset{\sim}{u}^n - \lambda \underset{\sim}{w}^n)$ is, from (4.23), a *quartic* polynomial in λ that we shall denote by $j_n(\lambda)$; λ_n is therefore solution of the *cubic* equation

$$(4.27)\qquad j_n'(\lambda) = 0 .$$

We shall use the standard Newton method to compute λ_n from (4.27), starting from $\lambda = 0$. The resulting algorithm is given by

$$(4.28)\qquad \lambda^0 = 0,$$

then for $k \geq 0$, *we obtain* λ^{k+1} *from* λ^k *by*

$$(4.29)\qquad \lambda^{k+1} = \lambda^k - \frac{j_n'(\lambda^k)}{j_n''(\lambda^k)} .$$

In our calculations, we always observed a very fast convergence of algorithm (4.28), (4.29). Once λ_n is known, we know $\underset{\sim}{y}^{n+1}$ since (from (4.22)) $\underset{\sim}{y}^{n+1} = \underset{\sim}{y}^n(\lambda_n)$.

If we count now the number of *Dirichlet systems* for $\alpha I - \nu\Delta$, to be solved at each iteration, we observe that we have to solve only *three* such systems, namely (4.24), (4.25) and then (4.17) (to obtain $\underset{\sim}{g}^{n+1}$) ; this number is *optimal* for a *nonlinear* problem since the solution of a *linear*

problem, by least squares-preconditioned conjugate gradient, requires the solution at each iteration of *two* linear systems associated to the preconditioning operator.

From the above remarks, it appears clearly that the practical implementation of algorithm (4.12)-(4.19) will requires an efficient (direct or iterative) elliptic solver, like one of those discussed in [27], [28]. As a final comment, we would like to mention that algorithm (4.12)-(4.19) (in fact its finite dimensional variants) is quite efficient ; when used in combination with the operator splitting methods of Sec. 3, three to five iterations suffice to reduce the value of the cost function by a factor of 10^4 to 10^6 ; however in view of other applications we are testing now some of those methods combining the features of conjugate gradients and quasi-Newton algorithms, such as the methods discussed in [29], [30] (the results recently obtained for the calculation of *transonic potential flows* containing *shocks*, look very promissing).

5. Solution of the "quasi" Stokes linear subproblems.

5.1. Generalities. Synopsis.

At each full step of the splitting methods discussed in Sec. 3.3 we have to solve one or two *linear problems* of the following type

$$(5.1)\qquad \begin{cases} \alpha \underset{\sim}{u} - \nu \Delta \underset{\sim}{u} + \underset{\sim}{\nabla} p = \underset{\sim}{f} \quad in\ \Omega, \\ \underset{\sim}{\nabla}.\underset{\sim}{u} = 0 \quad in\ \Omega, \\ \underset{\sim}{u} = \underset{\sim}{g} \quad on\ \Gamma \qquad (with \int_\Gamma \underset{\sim}{g}.\underset{\sim}{n}\, d\Gamma = 0), \end{cases}$$

where α and ν are two *positive* constants, and where $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are two *given* functions defined on Ω and Γ, respectively. We recall that if $\underset{\sim}{f}$ and $\underset{\sim}{g}$ are sufficiently smooth, then problem (5.1) has a *unique* solution in $V_g \times (L^2(\Omega)/\mathbb{R})$ (with V_g still defined by (4.5) ; $p \in L^2(\Omega)/\mathbb{R}$ means that p is defined only to within an arbitrary constant). We shall describe below two iterative methods for solving (5.1), quite easy to implement using finite element or spectral methods (more details are given in

[5] , [22] , together with convergence proofs ; more methods are discussed in [5]).

5.2. A conjugate gradient algorithm for solving (5.1).

A complete justification of the following algorithm is given in [5] ; we can say briefly that eliminating $\underset{\sim}{u}$ in (5.1), it appears that p is solution of a *linear* functional equation associated to an operator which is *self adjoint* and *strongly elliptic* from H onto H, where

$$H = \{q \mid q \in L^2(\Omega), \int_\Omega q \, dx = 0\} .$$

Such properties justify a conjugate gradient solution of (5.1) and lead to the algorithm below

$$p^0 \in L^2(\Omega), \quad \textit{given}, \tag{5.2}$$

$$\begin{cases} \alpha \underset{\sim}{u}^0 - \nu \Delta \underset{\sim}{u}^0 = \underset{\sim}{f} - \underset{\sim}{\nabla} p^0, \\ \underset{\sim}{u}^0 = \underset{\sim}{g} \quad \textit{on} \quad \Gamma, \end{cases} \tag{5.3}$$

$$g^0 = \underset{\sim}{\nabla} . \underset{\sim}{u}^0, \tag{5.4}$$

$$w^0 = g^0 . \tag{5.5}$$

Then for $n \geq 0$, p^n, g^n, w^n *being known, compute* p^{n+1}, g^{n+1}, w^{n+1} *as follows :*

Solve

$$\begin{cases} \alpha \underset{\sim}{\chi}^n - \nu \Delta \underset{\sim}{\chi}^n = -\underset{\sim}{\nabla} w^n, \\ \underset{\sim}{\chi}^n = \underset{\sim}{0} \quad \textit{on} \quad \Gamma, \end{cases} \tag{5.6}$$

$$\rho_n = \frac{\int_\Omega |g^n|^2 \, dx}{\int_\Omega \underset{\sim}{\nabla} . \underset{\sim}{\chi}^n w^n dx} , \tag{5.7}$$

$$p^{n+1} = p^n - \rho_n w^n, \tag{5.8}$$

$$\gamma_n = \frac{\int_\Omega |g^{n+1}|^2 \, dx}{\int_\Omega |g^n|^2 \, dx} \,, \tag{5.9}$$

$$w^{n+1} = g^{n+1} + \gamma_n w^n \,. \tag{5.10}$$

Do n = n+1, *go to* (5.6).

We observe that algorithm (5.2)-(5.10) requires the solution at each iteration of only *one* Dirichlet system, namely (5.6). It can be proved that

$$\lim_{n \to +\infty} \{\underset{\sim}{u}^n, p^n\} = \{\underset{\sim}{u}, p_o\} \tag{5.11}$$

where $\{\underset{\sim}{u}, p_o\}$ is that solution of (5.1) such that

$$\int_\Omega p_o \, dx = \int_\Omega p^o \, dx \,.$$

5.3. A second iterative method for solving (5.1).

This method is defined as follows (with r a *nonnegative* parameter) :

$$p^o \in L^2(\Omega), \text{ given,} \tag{5.12}$$

then for $n \geq 0$, *define* $\underset{\sim}{u}^{n+1}$ *and* p^{n+1} , *from* p^n *by*

$$\begin{cases} \alpha \underset{\sim}{u}^n - \nu \Delta \underset{\sim}{u}^n - r \, \underset{\sim}{\nabla}(\underset{\sim}{\nabla} . \underset{\sim}{u}^n) = \underset{\sim}{f} - \underset{\sim}{\nabla} p^n, \\ \underset{\sim}{u}^n = \underset{\sim}{g} \quad \text{on} \quad \Gamma \,, \end{cases} \tag{5.13}$$

$$p^{n+1} = p^n - \rho \, \underset{\sim}{\nabla} . \underset{\sim}{u}^n \,. \tag{5.14}$$

The above method is related to the *artificial compressibility* methods of Chorin and Yanenko since (5.14) can be considered as a discretization of

$$\frac{\partial p}{\partial t} + \underset{\sim}{\nabla} . \underset{\sim}{u} = 0$$

(ρ then plays the role of a time step).
About the convergence of (5.12)-(5.14) we should prove (see, e.g., [5, Chapter 7] for such a proof) the following

Proposition 5.1. : *Suppose that*

$$0 < \rho < 2\left(r + \frac{\nu}{N}\right) ; \tag{5.15}$$

we have then

$$\lim_{n \to +\infty} \{\underset{\sim}{u}^n, p^n\} = \{\underset{\sim}{u}, p_o\} \text{ strongly in } (H^1(\Omega))^N \times L^2(\Omega) \tag{5.16}$$

where $\{\underset{\sim}{u}, p_o\}$ *is the solution of* (5.1) *such that*

$$\int_\Omega p_o \, dx = \int_\Omega p^o \, dx .$$

Moreover the convergence is linear (i.e. $\|\underset{\sim}{u}^n - \underset{\sim}{u}\|_{(H^1(\Omega))^N}$ *and* $\|p^n - p_o\|_{L^2(\Omega)}$ *converges to zero as fast, at least, as geometric sequences).*

Remark 5.1: (*About the choice of* ρ *and* r) : We should use $\rho = r$ in practice, since it can be proved in that case that the convergence ratio of algorithm (5.12)-(5.14) is $0(r^{-1})$, for large values of r. In most applications taking $r = 10^2 \nu$ to $10^4 \nu$ we have a practical convergence of algorithm (5.12)-(5.14) in 3 to 4 iterations. There is however a practical upper bound for r, since, for too large values of r, problem (5.13) will be *ill-conditioned* and its practical solution sensitive to *round-off* errors.

Remark 5.2 : If $r = 0$, problem (5.13) reduces to the solution of N *uncoupled* (one for each component of $\underset{\sim}{u}^n$) *scalar Dirichlet problems* for $\alpha I - \nu\Delta$; if $r > 0$ the N components of $\underset{\sim}{u}^n$ are coupled by $\underset{\sim}{\nabla}(\underset{\sim}{\nabla}.\underset{\sim}{u}^n)$, making the solution of (5.13) much more costly. In fact, the elliptic operator

in the left hand side of (5.13) is very close to the *linear elasticity operator*, and close variants of it occur naturally in *compressible* and/or *turbulent* viscous flow problems.

Remark 5.3 : Other methods for solving (5.1) are discussed in [5], [31], [32].

6. Finite Element Approximation of the Time Dependent Navier-Stokes Equations.

We shall describe in this section a specific class of *finite element approximations* for the time dependent Navier-Stokes. Actually these methods which lead to *continuous approximations* for both pressure and velocity are fairly simple, and some of them have been known for years. They have been advocated for example by Hood and Taylor (see [33]). Other finite element approximations of the incompressible Navier-Stokes equations can be found in [1], [2], [4], [5], [34] (see also the references therein).

6.1. Basic hypotheses. Fundamental discrete spaces.

We suppose that Ω is a *bounded polygonal domain* of $\mathbf{R}^2$. With $\mathcal{T}_h$ a standard *finite element triangulation* of Ω, and h the maximal length of the edges of the triangles of $\mathcal{T}_h$, we introduce the following discrete spaces (with P_k = space of the polynomials in two variables of degree $\leq k$) :

$$H_h^1 = \{q_h \mid q_h \in C^0(\bar{\Omega}),\ q_h|_T \in P_1,\ \forall T \in \mathcal{T}_h\}\ , \tag{6.1}$$

$$V_h = \{\underset{\sim}{v}_h \mid \underset{\sim}{v}_h \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}),\ \underset{\sim}{v}_h|_T \in P_2 \times P_2,\ \forall T \in \mathcal{T}_h\}, \tag{6.2}$$

$$V_{oh} = \{\underset{\sim}{v}_h \in V_h,\ \underset{\sim}{v}_h = \underset{\sim}{0} \text{ on } \Gamma\} = V_h \cap V_o\ . \tag{6.3}$$

Two useful variants of V_h (and V_{oh}) are obtained as follows : either

$$V_h = \{\underset{\sim}{v}_h \mid \underset{\sim}{v}_h \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}),\ \underset{\sim}{v}_h|_T \in P_1 \times P_1,\ \forall T \in \tilde{\mathcal{T}}_h\}, \tag{6.4}$$

or (this space has been introduced in [35])

$$(6.5) \qquad V_h = \{\underset{\sim}{v}_h \mid \underset{\sim}{v}_h \in C^0(\bar{\Omega}) \times C^0(\bar{\Omega}),\ \underset{\sim}{v}_h|_T \in P^*_{1T} \times P^*_{1T},\ \forall T \in \mathcal{C}_h\}\ .$$

In (6.4), $\tilde{\mathcal{C}}_h$ is this triangulation of Ω obtained from $\mathcal{C}_h$ by joining the midpoints of the edges of $T \in \mathcal{C}_h$, as shown on Fig. 6.1 ; we have the same global number of unknowns if we use V_h defined by either (6.2) or (6.4), however the matrices encountered in the second case are more compact and sparse. In (6.5), P^*_{1T} is the subspace of P_3 defined as follows

$$(6.6) \qquad P^*_{1T} = \{q \mid q = q_1 + \lambda\phi_T,\ \textit{with}\ q_1 \in P_1, \lambda \in \mathbb{R},\ \textit{and}\ \phi_T \in P_3,$$
$$\phi_T = 0\ \textit{on}\ \partial T,\ \phi_T(G_T) = 1\}$$

where, in (6.6) G_T is the *centroid* of T (see Fig. 6.2 below). A function like ϕ_T is usually called a *bubble-function*.

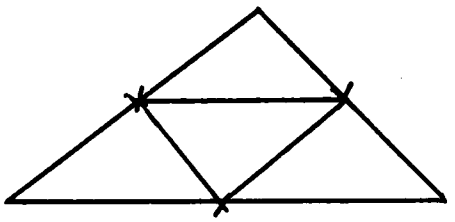


Figure 6.1.

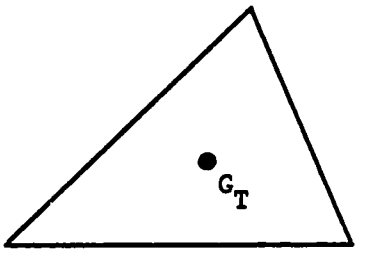


Figure 6.2.

6.2. Approximation of the boundary conditions.

If the boundary conditions are defined by

$$(6.7) \qquad \underset{\sim}{u} = \underset{\sim}{g} \quad \textit{on}\ \Gamma \quad \textit{with} \int_\Gamma \underset{\sim}{g}.\underset{\sim}{n}\, d\Gamma = 0,$$

it is of fundamental importance to approximate $\underset{\sim}{g}$ by $\underset{\sim}{g}_h$ such that $\int_\Gamma \underset{\sim}{g}_h.\underset{\sim}{n}\, d\Gamma = 0$. The construction of such $\underset{\sim}{g}_h$ is discussed in [5, Appendix 3].

6.3. Space approximation of the time dependent Navier-Stokes equations.

Using the spaces H_h^1, V_h and V_{oh} we approximate the time dependent Navier-Stokes equations as follows :

Find $\{\underset{\sim}{u}_h(t),\ p_h(t)\} \in V_h \times H_h^1$, $\forall t \geq 0$, *such that*

$$(6.8)\quad \begin{cases} \displaystyle\int_\Omega \frac{\partial \underset{\sim}{u}_h}{\partial t} \cdot \underset{\sim}{v}_h \, dx + \nu \int_\Omega \underset{\sim\sim}{\nabla u}_h . \underset{\sim\sim}{\nabla v}_h dx + \int_\Omega (\underset{\sim}{u}_h . \underset{\sim}{\nabla}) \underset{\sim}{u}_h . \underset{\sim}{v}_h \, dx \\ + \displaystyle\int_\Omega \underset{\sim}{\nabla} p_h . \underset{\sim}{v}_h \, dx = \int_\Omega \underset{\sim}{f}_h . \underset{\sim}{v}_h \, dx, \ \forall \underset{\sim}{v}_h \in V_{oh} , \end{cases}$$

$$(6.9)\quad \int_\Omega \underset{\sim}{\nabla} . \underset{\sim}{u}_h \, q_h \, dx = 0, \ \forall q_h \in H_h^1 ,$$

$$(6.10)\quad \underset{\sim}{u}_h = \underset{\sim}{g}_h \ \textit{on} \ \Gamma ,$$

$$(6.11)\quad \underset{\sim}{u}_h(x,0) = \underset{\sim}{u}_{oh} \ (\textit{with} \ \underset{\sim}{u}_{oh} \in V_h) ;$$

in (6.8)-(6.11), $\underset{\sim}{f}_h$, $\underset{\sim}{u}_{oh}$ and $\underset{\sim}{g}_h$ are convenient approximations of $\underset{\sim}{f}$, $\underset{\sim}{u}_o$ and $\underset{\sim}{g}$, respectively.

6.4. Time discretization of (6.8)-(6.11) by operator splitting methods.

We consider now a fully discrete version of scheme (3.40)-(3.42) discussed in Sec. 3.3 ; it is defined as follows (with Δt as in Sec. 3.3) :

$$(6.12)\quad \underset{\sim}{u}_h^o = \underset{\sim}{u}_{oh} ,$$

then for $n \geq 0$, *compute* (*from* $\underset{\sim}{u}_h^n$) $\{\underset{\sim}{u}_h^{n+1/2}, p_h^{n+1/2}\} \in V_h \times H_h^1$, *and then* $\underset{\sim}{u}_h^{n+1} \in V_h$, *by solving*

$$(6.13)_1\quad \begin{cases} \displaystyle\int_\Omega \frac{\underset{\sim}{u}_h^{n+1/2} - \underset{\sim}{u}_h^n}{\Delta t/2} . \underset{\sim}{v}_h \, dx + \frac{\nu}{2} \int_\Omega \underset{\sim\sim}{\nabla u}_h^{n+1/2} . \underset{\sim\sim}{\nabla v}_h \, dx + \int_\Omega \underset{\sim\sim}{\nabla} p_h^{n+1/2} . \underset{\sim}{v}_h \, dx = \\ = \displaystyle\int_\Omega \underset{\sim}{f}_h^{n+1/2} . \underset{\sim}{v}_h \, dx - \frac{\nu}{2} \int_\Omega \underset{\sim\sim}{\nabla u}_h^n . \underset{\sim\sim}{\nabla v}_h dx - \int_\Omega (\underset{\sim}{u}_h^n . \underset{\sim}{\nabla}) \underset{\sim}{u}_h^n . \underset{\sim}{v}_h dx, \ \forall \underset{\sim}{v}_h \in V_{oh} , \end{cases}$$

$$(6.13)_2\quad \int_\Omega \underset{\sim}{\nabla} . \underset{\sim}{u}_h^{n+1/2} \, q_h \, dx = 0, \ \forall q_h \in H_h^1 ,$$

$$(6.13)_3 \qquad \underset{\sim}{u}_h^{n+1/2} \in V_h,\ p_h^{n+1/2} \in H_h^1,\ \underset{\sim}{u}_h^{n+1/2} = \underset{\sim}{g}_h^{n+1/2} \ \textit{on}\ \ \Gamma,$$

and

$$(6.14) \qquad \begin{cases} \displaystyle\int_\Omega \frac{\underset{\sim}{u}_h^{n+1} - \underset{\sim}{u}_h^{n+1/2}}{\Delta t/2} \cdot \underset{\sim}{v}_h \, dx + \frac{\nu}{2} \int_\Omega \underset{\sim\sim}{\nabla u}_h^{n+1} \cdot \underset{\sim\sim}{\nabla v}_h dx + \\ + \displaystyle\int_\Omega (\underset{\sim}{u}_h^{n+1} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}_h^{n+1} \cdot \underset{\sim}{v}_h \, dx = \int_\Omega \underset{\sim}{f}_h^{n+1} \cdot \underset{\sim}{v}_h \, dx \\ - \frac{\nu}{2} \displaystyle\int_\Omega \underset{\sim\sim}{\nabla u}_h^{n+1/2} \cdot \underset{\sim\sim}{\nabla v}_h \, dx - \int_\Omega \underset{\sim}{\nabla} p_h^{n+1/2} \cdot \underset{\sim}{v}_h \, dx, \ \forall \underset{\sim}{v}_h \in V_{oh}, \end{cases}$$

$$(6.15) \qquad \underset{\sim}{u}_h^{n+1} \in V_h,\ \underset{\sim}{u}_h^{n+1} = \underset{\sim}{g}_h^{n+1} \ \textit{on}\ \Gamma,$$

respectively.

The same techniques apply to the space discretization of scheme (3.43)-(3.46) (see [5] for more details).
The solution of the various subproblems encountered at each step of (6.12)-(6.15) can be done by the discrete variants of the method discussed in Secs. 4 and 5, see,again, [5] for more details.

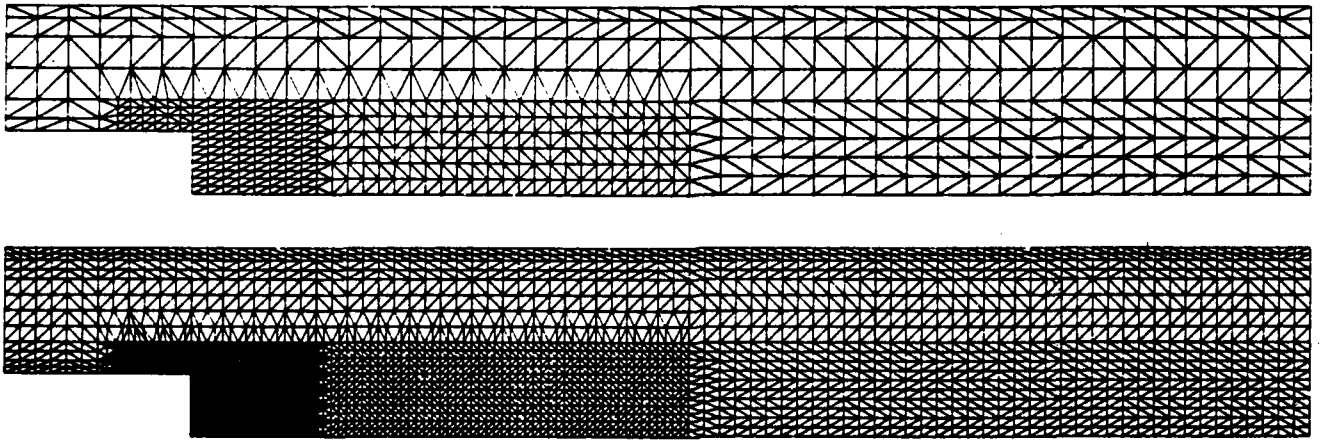
7. Numerical experiments.

We illustrate the numerical methods described in the above sections by the presentation of the results of numerical experiments where these methods have been applied to simulate some incompressible viscous flows of practical interest. All the calculations which follow have been done using the finite element method associated to the discrete spaces defined by (6.1), (6.3), (6.4) (some preliminary and promissing results have been obtained recently, using V_h defined by (6.5), (6.6)).

7.1. A first class of test problems

We consider the solution of the Navier-Stokes equations for the flow of an incompressible viscous fluid in the *channel with a step* of Figure 7.1. We have selected this problem since it is a quite classical and significant test problem for Navier-Stokes solvers (see [34] for

a comparison of various methods for solving this test problem). The finite element triangulations used for these calculations are shown on Figure 7.1. ; the *coarse* (resp. *fine*) one is used for approximating the *pressure* (resp. the *velocity*).



	$\mathcal{T}_h$	$\tilde{\mathcal{T}}_h$
Nodes	619	2346
Triangles	1109	4436
Cholesky's coefficients	21654	154971

Figure 7.1.

We have also indicated on Fig. 7.1., the number of nodes, triangles, and *nonzero* Cholesky coefficients of the matrix approximating $-\Delta$ (resp. $\alpha I - \nu\Delta$) on $\mathcal{C}_h$ (resp. $\tilde{\mathcal{C}}_h$). The methods described in the above sections have been applied to compute the *steady state solutions* of (2.1), (2.2) for the following boundary conditions

(7.1) $\begin{cases} \underset{\sim}{u} \textit{ satisfies Poiseuille velocity profiles at the entrance} \\ \textit{and exist of the channel and is equal to } \underset{\sim}{0} \textit{ elsewhere on } \Gamma. \end{cases}$

We have taken Re = 100 and 191. The numerical results agree quite well with those in [34], [36] and show a clear superiority of the schemes derived from (3.43)-(3.46), over the schemes derived from (3.40)-(3.42). For the scheme (3.43)-(3.46) we have tested $\theta = .25$ and $\theta = 1-\sqrt{2}/2$ (for the same Δt) ; the convergence to the steady state is faster with the

second value of θ (for this class of problems at least). We have shown on Fig. 7.2 (resp. Fig. 7.3) the *stream* lines (resp. the *isobar* lines), of the steady state solutions corresponding to Re = 100 and Re = 191. We observe that the size of the *recirculation region* increases with Re.

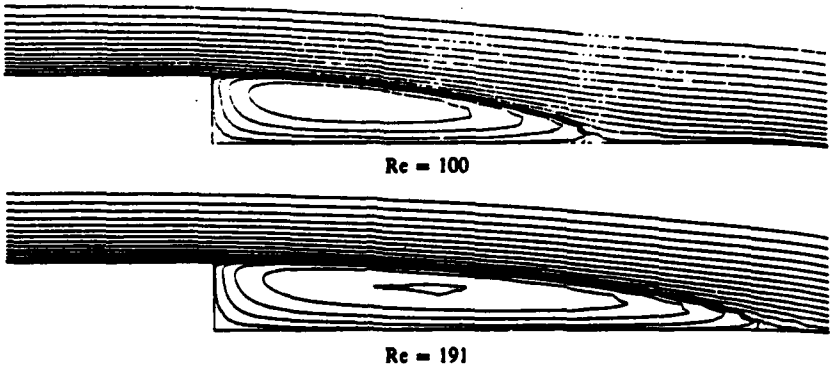


Figure 7.2.
Stream lines

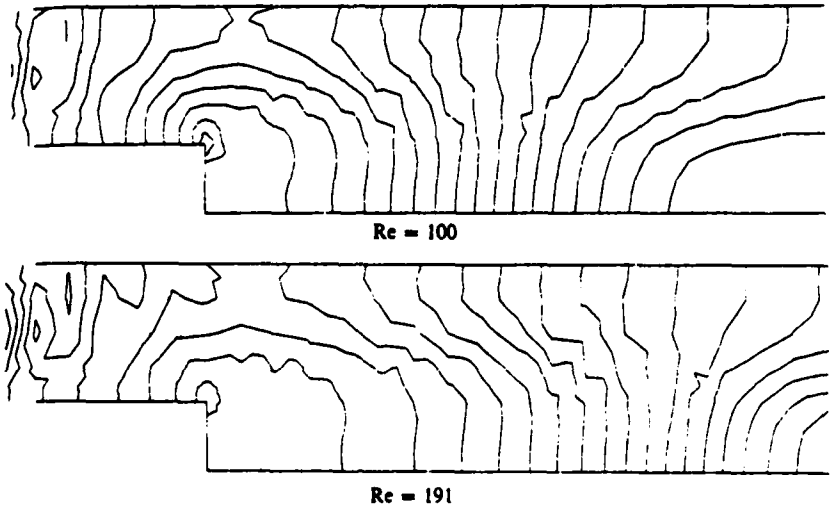


Figure 7.3.
Isopressure lines

7.2. A second class of test problems

The second test problem that we considered is much more complicated than the first one, since it concerns the simulation of an incompressible viscous flow around and inside a (two-dimensional) nozzle at high incidence (40 degrees), and at Re = 750 (the characteristic lengtn being the distance between the walls of the nozzle). We used the same kind of finite element approximation than in Sec. 7.1. Figures 7.4, 7.5 show the details of the triangulations $\mathcal{C}_h$ and $\tilde{\mathcal{C}}_h$, respectively, close to the air intake, Figures 7.6-7.10 show the stream lines and the vortex patern of the flow at t = .0, .2, .4, .6, .8, the initial velocity being associated to the corresponding steady Stokes flow, and a suction phenomenon being simulated inside the nozzle.

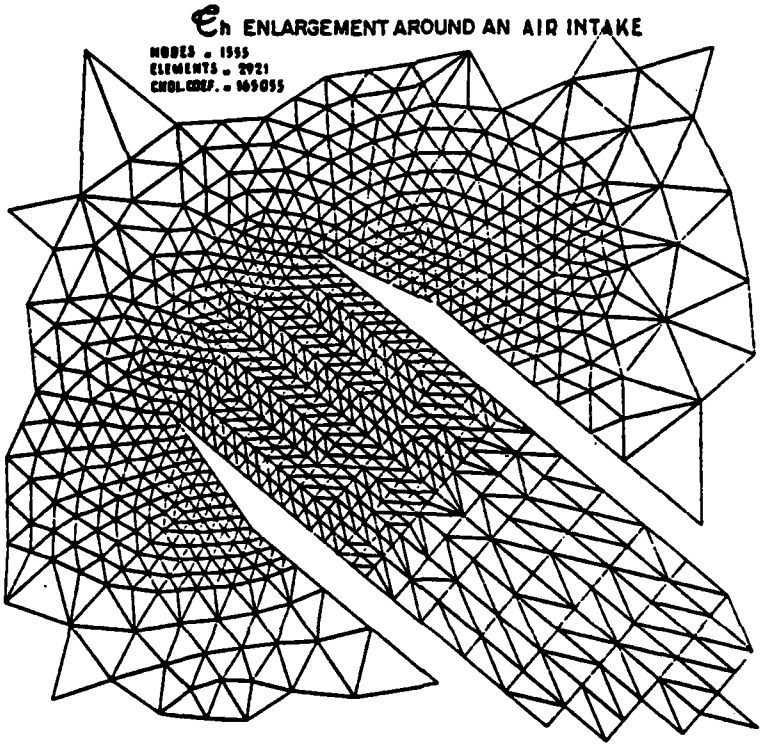


Figure 7.4.
Pressure grid

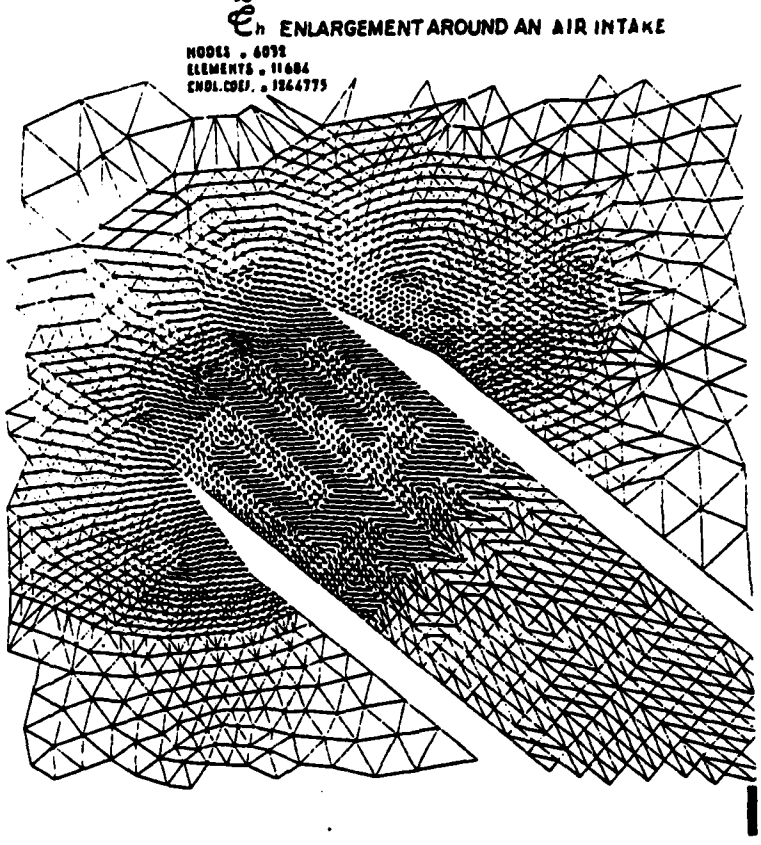


Figure 7.5.

Velocity grid

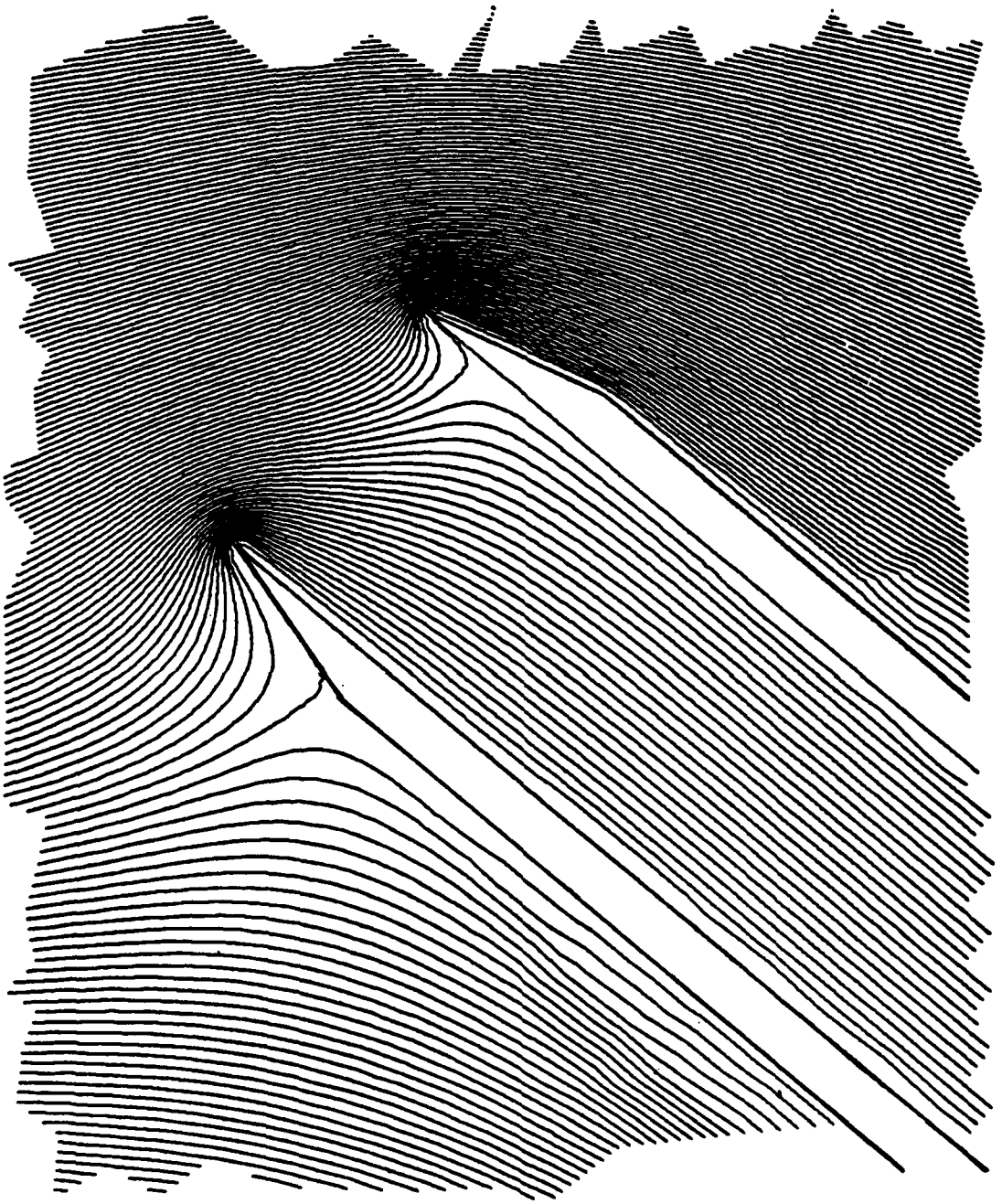


Figure 7.6.

Re = 750, t = .0

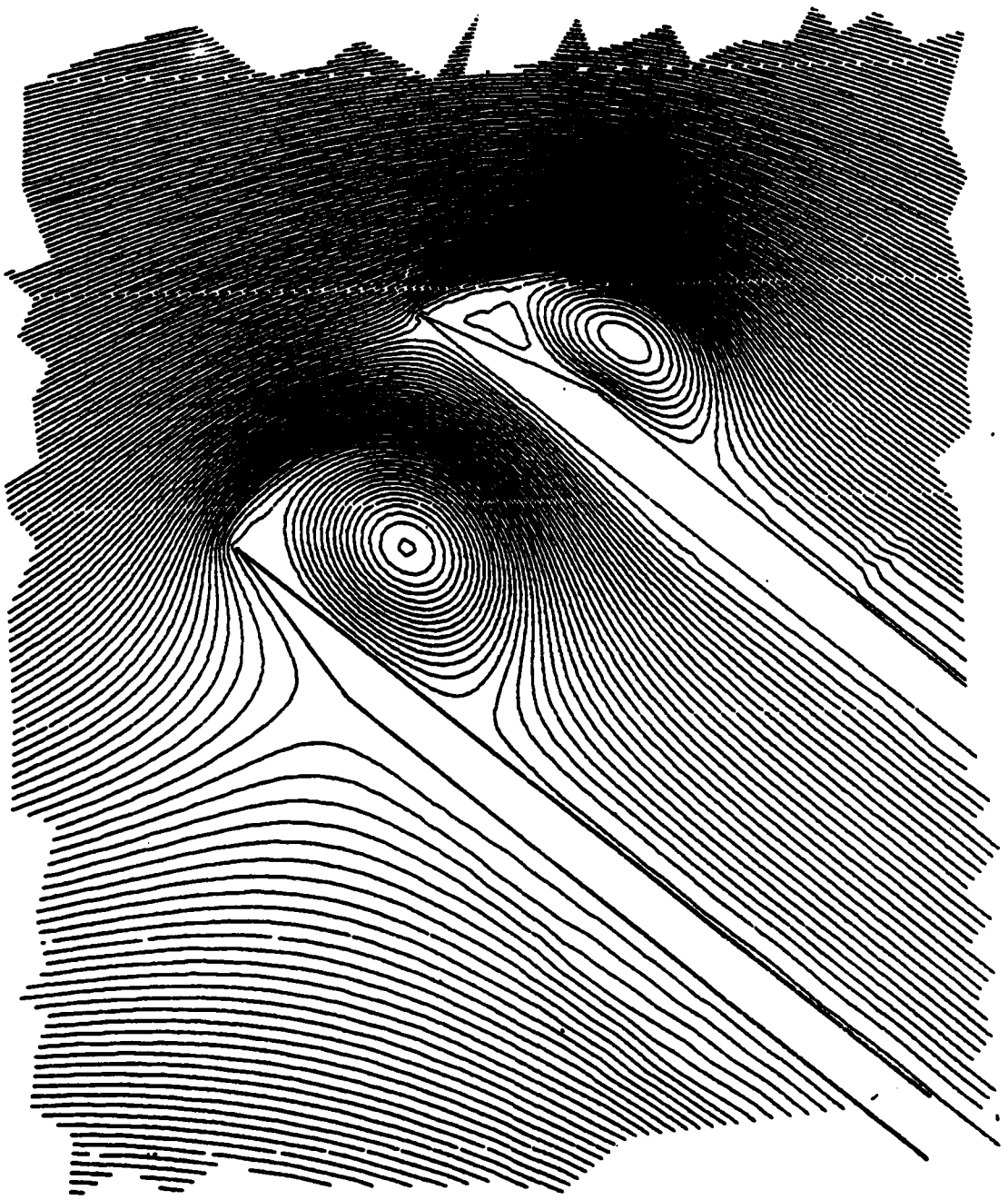


Figure 7.7.

Re = 750 ; t = .2

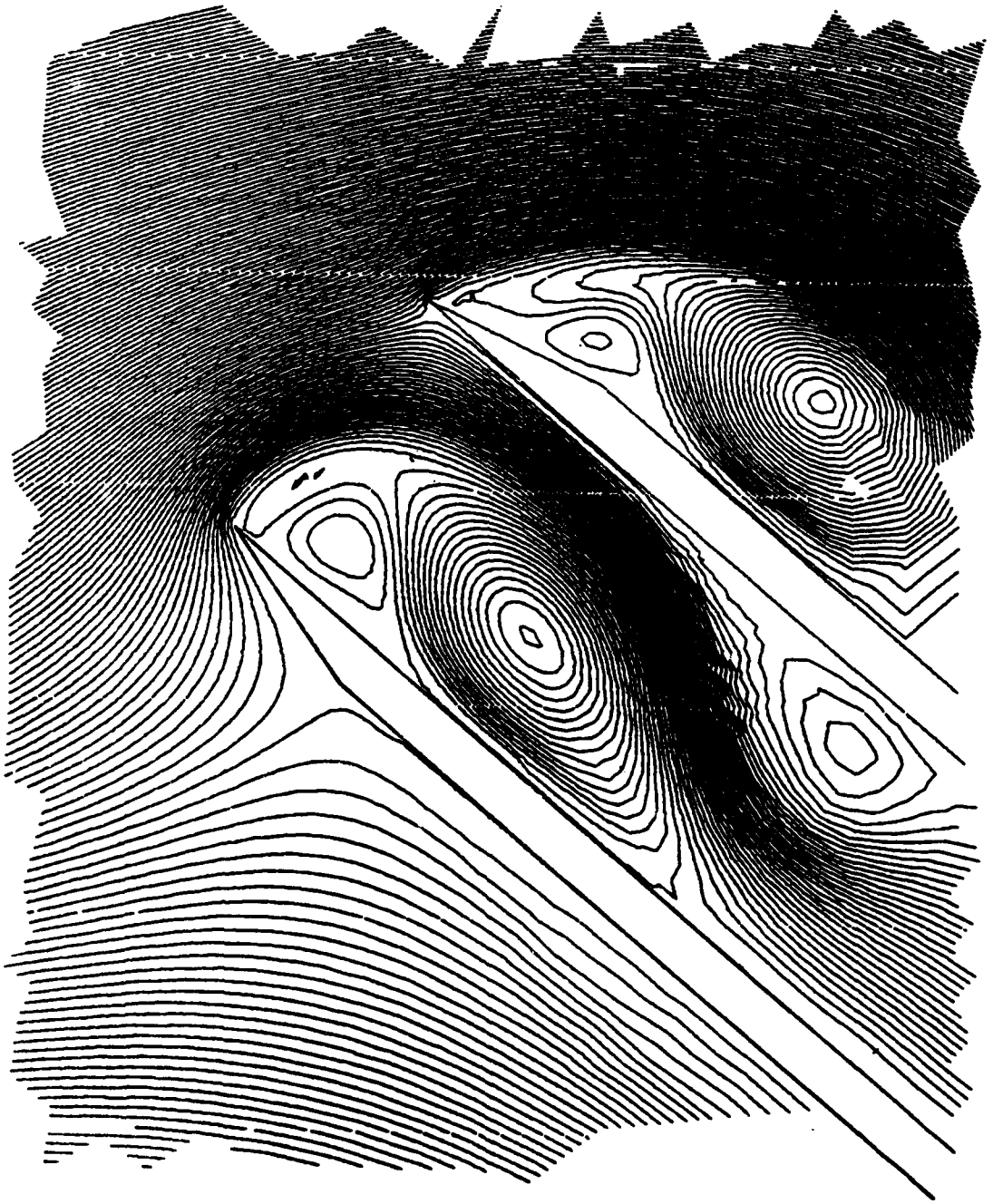


Figure 7.8.

Re = 750 ; t = .4

Figure 7.9.

Re = 750 ; t = .6

Figure 7.10.

Re = 750 ; t = .8

8. Conclusion

We have presented in this paper numerical methods for solving the time dependent incompressible Navier-Stokes equations. One of the key ingredient to make the solution process faster, and less demanding from the computer storage point of view, was definitely the use of *operator splitting methods* in order to decouple the two main difficulties of the problem, namely the *nonlinearity* and the *incompressibility*. Others splitting methods for the incompressible Navier-Stokes equations are discussed in [37] - [40], [1], [17]. Another important reference for operator splitting methods is [41].

Application of these splitting methods to *nonlinear eigenvalue problems* will be discussed in a forthcoming paper.

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Splitting methods provide efficient tools for solving linear and nonlinear time dependent problems modelled by partial differential equations. In this report we discuss the numerical solution of the Navier-Stokes equations for incompressible viscous fluids by such methods. The splitting permits decoupling the two main difficulties in the problem, namely the nonlinearity and the incompressibility. Actually these splitting methods have a broad range of applicability and can be applied for example, to the solution of eigenvalue problems.

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